

Notes on Geometric Measure Theory Applications to Image Processing; De-noising, Segmentation, Pattern, Texture, Lines, Gestalt and Occlusion.

Simon P Morgan
University of Minnesota, morgan@math.umn.edu

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Abstract

Regularization functionals that lower level set boundary length when used with L^1 fidelity functionals on signal de-noising on images create artifacts. These are (i) rounding of corners, (ii) shrinking of radii, (iii) shrinking of cusps, and (iv) non-smoothing of staircasing. Regularity functionals based upon total curvature of level set boundaries do not create artifacts (i) and (ii). An adjusted fidelity term based on the flat norm on the current (a distributional graph) representing the density of curvature of level sets boundaries can minimize (iii) by weighting the position of a cusp. A regularity term to eliminate staircasing can be based upon the mass of the current representing the graph of an image function or its second derivatives. Densities on the Grassmann bundle of the Grassmann bundle of the ambient space of the graph can be used to identify patterns, textures, occlusion and lines.

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1 Introduction

This is an informal discussion paper suggesting possible use of ideas in geometric measure theory from minimal surfaces, rather than BV, for image processing. In particular we investigate the use of rectifiable currents in various spaces to represent signals and images with the flat norm, the natural topology for currents. We use oriented Grassmann bundles as they are the spaces that represent both direction and position. As this is an exploratory discussion paper there are no proofs, references in image processing, literature review or reports on computational results at this stage. It is intended that these should be added to subsequent versions of this paper, along with pictures and where errors and omissions will hopefully be corrected.

Notation 1 $A \subset \mathbb{R}^2$ is the domain of the image.

$A \times [0, 1]$

$G_A = A \times S^1$ denotes the oriented Grassmann bundle of A . This implicitly involves a choice of basis.

S^2 will be used to denote the oriented Grassmannian of \mathbb{R}^3

S^4 will be used to denote the oriented Grassmannian of \mathbb{R}^5

$[f]$ is the rectifiable current represent the graph of f

$\partial[f]$ is the rectifiable current representing boundary of graph of f which will include discontinuities.

$M(T)$ is the mass of current T .

$\mathcal{F}(T)$ is the current flat norm of T .

2 Examples and philosophy

	Mass	Curvature
Regularity for de-noising and segmentation	$M([f]) + M(\partial[f])$ (R_4) prevents staircasing on $[f]$ and $\partial[f]$	Unsigned R_1, R_2, R_3, R_5 prevents staircasing, no shrinking radii or smoothing
Fidelity for de-noising and segmentation	$L^1(F_1)$ prevents average values drifting	Flat norm on signed curvature prevents cusp drift
Pattern Texture Lines Gestalt Occlusion	Graph density and spatial frequencies	Densities and spatial frequencies on $A \times S^2$ and $A \times [0, 1] \times S^2 \times S^4$

The aim is to remove noise from a signal g by finding a de-noised image f . Noise is small scale and randomly structured thus having high curvature. We can approach this with regularity terms that assign a cost to irregularity, and a fidelity term which assigns a cost to deviation from a signal.

We want the regularity terms to be scale invariant, so small versions of a shape cost the same as large versions, but we want fidelity to not be scale invariant so we allow small versions to be eliminated for less than large versions, for the same gain in regularity.

For the regularity we choose total curvature for its scale invariance. For fidelity we chose the flat norm on curvature for it gets proportionately cheaper as scale becomes smaller.

An obvious question is why not just use the standard L^1 norm for fidelity. This will work fine where level sets are always smooth, but will introduce distortion on cusps. Therefore we use curvature on the fidelity term with the flat norm.

2.1 Large bumps (object) vs small bumps (noise)

The flat norm on any 1-current corresponding to a bump (level set reduced boundary curvature density) will reduce more than linearly under homothetic shrinking.

$$F(\lambda_{\frac{1}{c}} T) \leq \frac{1}{c} F(T)$$

Equality for all c only occurs in special geometric situations with straight line segments. Therefore noise will follow the strict inequality as $t < 1$ in the following:

$$F(\lambda_{\frac{1}{c}\#}T) = F(T)\{\frac{1}{c}t + (1-t)\frac{1}{c}\} < \frac{1}{c}F(T)$$

Now we use flat norms for fidelity terms but not for regularity. This means removal of small objects cheaper than removing large ones for the same gain in regularity.

2.2 Fine undulations (noise) vs coarse undulations (objects)

Regularity should be unsigned to prevent cancellation, making undulations expensive, whereas signed measure for fidelity enables cancelling on flat norm.

2.3 Line drawings

In line drawings the level set boundaries are very close and can cancel as their orientations as currents are opposite. However for low curvatures there is a low regularity term associated with the curves. There will be critical line thickness if we represent it as level sets. If however we represent it as a 2 current of the graph, then we can consider the density weighting to be by Gaussian curvature. This will give low density to cylindrical regions or product like regions which is what the graph of a line is like.

Another possibility is to switch to representing a thin line as a 1 current, rather than a thick region whose level set boundaries are 1 currents. A way to detect such lines is to see that the level set boundaries cancel almost completely under the flat norm but do not under the mass norm, and the mass is concentrated in certain directions.

2.3.1 Gestalt and Occlusion

By considering lines or level set boundaries as 1 currents we are naturally considering measures in the oriented Grassmann bundle of A , G_A , that is $A \times S^1$. Consider a projection p of this space onto $L \times S^1$, where P projects A orthogonally onto L . We can allow the measure in G_A to be pushed forward under p . The measure in points in S^1 , above point y in L , corresponding to the directions perpendicular to L in A , will indicate whether or not there was any measure on the line perpendicular to L passing through y . Now we can detect mass on lines in A .

The flat norm can also be used for this purpose. Lift the 1 currents in A to G_A . Place a cost penalty on boundary and isolated direction change of lines in A . Isolated direction changes in A correspond to currents in G_A , moving in the directions of the local basis of S^1 . We can simply increase the current density if it moves in those directions. This has the practical effect of filling in broken straight lines that correspond to occlusion or to continuation in patterns. Consider a road intersection from above. The road is one tone and the sides another. Eight straight line segments meet in pairs at 90 degrees. Under the flat norm on the currents in G_A , the colinear lines will be filled in. They do not

interact or interfere with each other. There is no need to decide if one road is occluding the other.

2.4 Texture and pattern

Using the projection above for occlusion, we can identify parallel or nearly lines in patterns and texture. This can be done by taking the projection of the mass of the currents in G_A , and determining which directions have the mass, and what the spatial frequency of the mass is along lines such as L .

Doing this on bundles such as $A \times [0, 1] \times S^2 \times S^5$, the oriented Grassmann bundle of the oriented Grassmann bundle, will give more information.

3 Level Set Total Curvature Regularization terms

The type of curvature here is the curvature of boundaries of levels sets as they are easy to define in terms of functions on domains. Instead it is possible to consider the curvature of the graph as an embedded submanifold. This is mentioned below in another section.

We are trying to optimize by varying f for a given g , the energy:

$$E(f, g) = \gamma_1 R_1(f) + \gamma_2 R_2(f) + \gamma_3 R_3(f) + \gamma_4 R_4(f) + \gamma_5 R_5(f) + \gamma_6 F_1(f, g) + \gamma_7 F_2(f, g)$$

3.0.1 The C^2 term (R_1)

Say we have an a.e. C^2 candidate function f for de-noising. We can define its total curvature as.

$$R_1(f) = \int |\nabla f| \left| \left(\frac{\nabla f}{|\nabla f|} \right)_{J \frac{\nabla f}{|\nabla f|}} \right| dx dy$$

Here $J \frac{\nabla f}{|\nabla f|}$ is a unit vector perpendicular to ∇f in the direction given by rotation by $\pi/2$.

3.1 The discontinuous case (R_2)

Say we now allow f to have a 1- dimensional rectifiable set upon which it is discontinuous. as a graph. Also now we can define the 2-dimensional rectifiable current $[g]$ as the current representing the set $F = \{(x, y, g(x, y))\}$. Now we can define $\partial[g]$ as the one dimensional rectifiable current representing $\partial\{(x, y, g(x, y))\}$ the boundary of F if it is embedded surface in \mathbb{R}^3 . There may be isolated points where the graph is not continuous, but these and any set of \mathcal{H}^2 measure zero will be ignored by the currents.

$F_s = p\#(Supp\partial[g])$ where p is projection onto A

$$R_2(f) = \int_{F_s} |\Delta\theta| dH^0 + \int_{F_s} |\theta'| dH^1$$

The first term corresponds to angles in the discontinuity, the second to smooth curves. θ is the angle of the tangential direction to F_s

3.1.1 Generating a discontinuous graph from a signal

If the gradient of the signal is high and does not vary much over a more or less convex region of the domain, then the regularity may increase greatly by introducing a discontinuity. This can be done by choosing a threshold value for the gradient and then a minimum length and height of discontinuity. The check that the cut really does reduce regularity costs without adding to much fidelity cost. Here the choice of unit vector scale will determine the relative cost of edge length for volume are of L^1 fidelity.

3.2 The continuous, non- C^1 case(R_3)

There may be polyhedral types of edges on the graph that can be represented as having curvature. This can given as

$$R_3(f) = \int |\Delta\theta| dH^1$$

Note that we need $\Delta\theta = 0$ almost everywhere.

4 Fidelity terms (F_1, F_2)

The usual fidelity term given a signal g and candidate de-noised function f is

$$F_1(f, g) = \int |f - g| dx dy$$

We add a cusp stability term based on the flat norm of 1-currents representing curvature level set density.

$$F_2(f, g) = \mathcal{F}(F_\theta - G_\theta)$$

$$\text{Where } F_\theta = \left[|\nabla f| \left(\frac{\nabla f}{|\nabla f|} \right) \right]_{J \frac{\nabla f}{|\nabla f|}} + [F_s]\theta' |f_1 - f_2| + [S]\Delta\theta |f_1 - f_2| \quad (\text{think}$$

of lift to $\mathbb{R}^2 X S^1$, but unlifted it is still a current, just not rectifiable)

Here $[F_s]$ and $[S]$ are 1-currents.

As one currents $F_\theta - G_\theta = R + \partial T$ where T is a 2 current

$$\mathcal{F}(F_\theta - G_\theta) = \min_{R, T} (M(R) + M(T)) \text{ where } F_\theta - G_\theta = R + \partial T$$

This is hard to compute and involves a transport problem.

An approximate way to compute the flat norm is to take $\sup_{\phi} (F_\theta - G_\theta)(\phi)$

where $|\phi| \leq 1$, and $\|d\phi\| = |d\phi| \leq 1$. This will need to be done under a suitable scaling.

4.1 Transportation cost.

The second term curvature measure fidelity is evaluated in terms of the flat norm. Note that is signed. This means that undulations on a small scale will cancel, and this will give a huge benefit to regularity. However if we have cusps that create large fidelity terms then they will not move too far without a big fidelity penalty.

5 Anti-staircasing regularity (R_4, R_5)

$R_4(f) = M([f]) + M(\partial[f])$ as a regularity term will remove staircasing but may re-introduce distortion. A better approach may be to take components of curvature in vertical directions i.e. f_{xx}, f_{yy}, f_{xy} . Call this $R_5 = \int |f_{xx}|^2 + |f_{yy}|^2 + |2f_{xy}|^2 dx dy$

6 Scaling and the flat norm vs regularity.

Here we see that how the length of unit vector is assigned to a signal domain will determine the scale amount of denoising. This works because the flat norm works across two different dimensional measures. The length of the unit vector determines the relationship between these measures.

6.1 Uniform packing of circles

Say we have simulated noise of the form of n^2 discs of radius $1/n$, and the discs are $2n$ apart. As n varies, this acts like a homothety on a pattern. If we take as fidelity the 1-current $\theta'[B]$ representing the curvature weighted boundary $[B]$ of these discs we find the flat norm $\mathcal{F}(\theta'[B])$ is

$$n^2 \min(2\pi, \frac{\pi}{n^2}) = \pi \quad \text{for } n > \frac{1}{\sqrt{2}}$$

However if we take absolute value of curvature weighted boundary as regularity cost we get $n^2 2\pi$

Clearly eliminating the noise on a scale below where $\pi < n^2 2\pi$ will reduce overall energy (Regularity-fidelity).

So for $n > \frac{1}{\sqrt{2}}$, this will occur.

As a result the scale on which the length of unit is set is the order of size where denoising will eliminate signal features of this form.

6.2 Edge oscillations

Say we have a saw tooth with angle θ from the mean boundary direction. Let $\frac{2}{n}$ be the period of the sawtooth.

Fidelity is $n \cdot \min(2\theta, \frac{\theta}{n \cos \theta}) = \frac{\theta}{\cos \theta}$ for large $n \dots n > 1/2 \cos \theta$

Regularity is $n\theta$

So for $n > 1/\cos \theta$, the saw tooth will be wiped out in optimization. Again how the unit is scaled will determine how much smoothing occurs.

6.3 Thin ellipses

As 1-currents these will self cancel in the flat norm. We can model this with semicircles of radius $1/n$.

The regularity is π , the flat norm is $\frac{\pi}{2n^2} + \frac{2}{n}$. So again the scaling of the unit vector in the domain determines the scale of the de-noising.

7 Representation and computation

7.1 Representing a known current

Given a 1 current S in \mathbb{R}^2 , we have the following representation $\{(x, y, a, b) : x, y \in n/m, S \text{ is given by } a(x, y)dx + b(x, y)dy\}$

Similarly a 2 current T in \mathbb{R}^3 can be represented by $\{(x, y, z, a, b, c) : x, y, z \in n/m \text{ where } T \text{ is given by } a(x, y)dydz + b(x, y)dzdx + c(x, y)dxdy\}$

To find a discrete set of values for a given current:

- (i) find a polyhedral approximation of the support set by straight line segments or triangles.
- (ii) for each segment or triangle find the average density and the direction vector coefficients (a, b) or (a, b, c)
- (iii) assign position coords for the midpoint of the segment or triangle and combine with direction coeffs giving (x, y, a, b) or (x, y, z, a, b, c)

7.2 Building a current from signal data

Say we have a level set we wish to represent as a 1-current.

We need positions of the level set and directions, but we also need density that we are going to take as θ' . If all the directions are represented as unit vectors \mathbf{T} , and we move along the path of the boundary at unit speed, $\frac{d\mathbf{T}}{dt}$ is $\frac{d\theta'}{dt}$.

7.3 Optimizing energy of the represented current

Now when it comes to computing gradient descent of overall cost functional we need to identify those parts of S or T that have a high regularity cost for a relatively low flat norm penalty.

Regularity cost on a region is given by $\sum \sqrt{a^2 + b^2}$

Flat norm fidelity penalty on a region is given by $\max_{\theta} (\cos(\theta) \sum a + \sin(\theta) \sum b)$. By

this we mean what is the contribution to $E(f, g)$ if $f = \text{constant}$ locally instead of having the local features of g in the region being summed over.

The flat norm of the difference between an f and g if f and g vary by diffeomorphism on the domain A , is that it is the sum over all features of the minimum of twice the current mass associated to the feature or the current mass times the distance moved.

The scale of de-noising determines the size of the region to be taken for the comparison.

If fidelity penalty is too low, then start moving to reduce regularity cost by some method

Possible methods

- 1 reduce length of boundary
- 2 on a semi global scale reduce curvature possibly by using θ information above.

8 Geometric Measure Theory Books

Introductions are given in:

Morgan: Frank Morgan; *Geometric measure theory: a beginner's guide*. 3rd Ed. (2000) This is really a guide and contains many pictures, examples, brief statements of results and can really help readers who are working through a deeper text.

Hardt, R. and Simon, L. *Seminar on Geometric Measure Theory*. Birkhauser (1986)

A deeper treatment is given in:

LY: Lin and Yang: *Geometric measure theory: an introduction*. 1st Ed. International Press. Boston. (series in advanced Mathematics Volume 1)

Lectures on Geometric Measure Theory by Leon Simon. available from the Centre for Mathematical Analysis, Australian National University, Volume 3, 1983

The main reference text for this field is *Geometric Measure Theory*: Federer. 1969 (reprinted by Springer verlag 1996)